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Higher-order JWKB approximations for radial problems: II. The quartic oscillator

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Abstract. The modified effective potential method for treating radial problems in the JWKB approximation is applied to the quartic oscillator defined by the potential $V(r)=r^4$. The JWKB quantisation condition for the energy W is shown to be expressible as $(2n_r+1)\pi = AW^{3/4}+B+CW^{-3/4}+DW^{-9/4}+O(W^{-15/4})$. The l-dependent coefficients A, B, C and D are determined exactly by taking into account contributions from *all* orders. On inversion, the above series yields an explicit analytic formula for the energy levels. This formula is easily generalised to d dimensions, and found to reproduce known numerical eigenvalues extremely well.

1. Introduction

It has been known for a long time that in a JWKB analysis of the radial Schrödinger equation, correct results are not obtained if one merely applies the one-dimensional JWKB formalism, taking as an effective potential the sum of the true potential $V(r)$ and the centrifugal barrier $l(l+1)/2r^2$. In such a treatment, one finds that the JWKB wavefunction has a behaviour near $r=0$ that is not only different from that of the exact wavefunction, but also different in different orders of the approximation. Now, near the origin the exact wavefunction goes as r^{l+1}, provided the potential $V(r)$ is less singular at $r=0$ than r^{-2}. This correct dependence can be obtained, in the lowest order of approximation, if one modifies the effective potential by replacing $l(l+1)$ by $(l+\frac{1}{2})^2$. This is the well known Langer–Kemble modification. What is not well known is that this modification is correct only for the lowest order. When one considers higher orders of the approximation, it becomes necessary to make further modifications. In a recent work we have considered this problem in some detail, and shown how to determine, in any order of the approximation, the modification of the effective potential that will lead to a JWKB wavefunction with the correct behaviour near the origin (Seetharaman and Vasan 1984, hereafter referred to as I). When applied to the isotropic harmonic oscillator and the Coulomb problems, our modified effective potential method yields, in both cases, the exact spectrum in every order of the approximation.

In this work we apply the modified effective potential method to the three-dimensional quartic oscillator with the potential $V(r)=r^4$, and carry out systematically a higher-order JWKB analysis of the eigenvalue problem up to the fourth order. There is no difficulty in principle in extending the analysis to still higher orders. The energy eigenvalues W are determined by the quantisation condition which is expressible as

$$(2n_r+1)\pi = f(W, l)$$

where n_r and l are quantum numbers, and f is a sum of integrals which contain W as a parameter. These JWKB integrals arise in different orders, and can all be evaluated in terms of complete elliptic integrals. Summing the contributions to f from all orders we show that f can be expanded as

$$f = AW^{3/4} + B + CW^{-3/4} + DW^{-9/4} + \mathrm{O}(W^{-15/4}). \tag{1.1}$$

All the coefficients in this expansion except A receive contributions from every order of the approximation. A is completely determined by the lowest order. We show that the contributions to B, C and D coming from *all* orders can be easily taken into account, which leads to the exact determination of these coefficients. We then invert the above expression, neglecting terms lower than $W^{-9/4}$, and obtain an explicit analytic expression for W as a function of n_r and l. The formula for the energy levels thus obtained is found to reproduce known numerical results extremely well. With the simple replacement $l \to l + \frac{1}{2}(d-3)$, the formula gives the energy levels of the quartic oscillator in $d(>1)$ dimensions.

With reference to our results for the r^4 potential, the following remarks may be noted. The present analysis is a distinct improvement over our earlier method for determining the energies, which was based on the lowest-order JWKB approximation (Seetharaman *et al* 1982). When l is set equal to zero, our expressions coincide with those of Pasupathy and Singh (1981) who have developed, for S-waves, a method for calculating higher-order JWKB integrals and given explicit expressions, up to the second order, for power law potentials. Our formula is as accurate as (and for many levels slightly better than) the empirical formula of Mathews *et al* (1982) and has the additional virtue of being derivable from theory. For the quartic oscillator in d dimensions, our results are decidedly superior to the ones quoted by Hioe (1978) whose formula corresponds to retaining only A and B in (1.1).

2. Quantisation formula for radial problems

For a particle of unit mass moving in a spherically symmetric potential $V(r)$, the quantisation condition for the energy W is given by (with $\hbar = 1$)

$$2\pi n_r = \oint \sum_{n=0}^{\infty} (-\mathrm{i})^n y_n \, \mathrm{d}r \tag{2.1}$$

where n_r is a non-negative integer (the radial quantum number). This is a radial generalisation of the one-dimensional formula of Dunham (1932). (For application of Dunham's formula to one-dimensional potentials see Bender *et al* (1977).) The y_n's are the different terms in the JWKB expansion of the radial wavefunction u:

$$u = \exp\left(\mathrm{i}\int y \, \mathrm{d}r\right) = \exp\left(\mathrm{i}\int \sum_{n=0}^{\infty} (-\mathrm{i})^n y_n \, \mathrm{d}r\right). \tag{2.2}$$

The expression for y_0 is

$$y_0^2 = 2(W - V_{\mathrm{eff}}) \tag{2.3a}$$

where the effective potential V_{eff} is defined by

$$V_{\text{eff}} = V(r) + L^2/2r^2. \tag{2.3b}$$

The other y_n's are to be found from the recurrence relation

$$2y_0 y_n + \mathrm{d}y_{n-1}/\mathrm{d}r + \sum_{m=1}^{n-1} y_n y_{n-m} = 0, \qquad n \geq 1. \tag{2.4}$$

As shown in I, L^2 is a parameter dependent on l which takes different values in different orders of the JWKB approximation. For potentials $V(r)$ satisfying $r^2\, V(r) \to 0$ as $r \to 0$, the values to be chosen for L in different orders of the approximation are as follows. In the lowest order (zeroth plus first) $L = l + \frac{1}{2}$, while it is the root of the equation $x + 1/8x = l + \frac{1}{2}$, and $x + 1/8x - 1/128x^3 = l + \frac{1}{2}$, in the second and fourth orders of approximation respectively. When *all* orders of the JWKB approximation are taken into account, its value is given by $L^2 = l(l+1)$. We assume that $V(r)$ satisfies the above condition at the origin and that the problem admits only two classical turning points for physical values of W. These turning points are branch points of y_0 and hence of y. We shall take the r plane to be cut along the real axis between the two turning points. The contour in (2.1) goes around the two branch points and encloses the cut. For definiteness, we shall take the contour to be traversed in the clockwise direction. We must then choose the branch of y_0 that is positive real on the upper lip of the cut. Thus

$$y_0 = +(2W - 2V(r) - L^2/r^2)^{1/2}. \tag{2.5}$$

Having determined y_n's (using (2.5) and (2.4)), we can calculate the integrals on the RHS of (2.1) in any given order of approximation. Clearly, to calculate the RHS of (2.1) in the fourth order of the approximation, it is enough to retain only terms up to $n = 4$ in it. Of these terms, it is easy to see that there is no contribution from y_3, since it can be expressed as a total derivative and therefore integrates to zero around the closed contour. Further, y_1 is a logarithmic derivative, and its integral is easily evaluated

$$\oint y_1\, \mathrm{d}r = -\pi \mathrm{i}, \tag{2.6}$$

independent of $V(r)$. Therefore the quantisation condition (in the fourth order) can be written as

$$\pi(2n_r + 1) = \oint (y_0 - y_2 + y_4)\, \mathrm{d}r \equiv I_0 + I_2 + I_4. \tag{2.7}$$

It is obvious that I_0 is given by

$$I_0 = \oint y_0\, \mathrm{d}r = \sqrt{2} \oint Q^{1/2}\, \mathrm{d}r \tag{2.8}$$

with

$$Q = W - V_{\text{eff}}. \tag{2.9}$$

Expressions for y_2 and y_4 can be obtained from (2.4). Noting that total derivatives in the integrands integrate to zero over the closed contour, considerable simplification of the integrals can be effected by dropping such terms in y_2 and y_4. The resulting expressions are found to be given by

$$I_2 = -\tfrac{1}{96}\sqrt{2}\oint Q^{-3/2}Q''\,\mathrm{d}r, \tag{2.10}$$

$$I_4 = -(\sqrt{2}/6144)\oint (7Q^{-7/2}(Q'')^2 - 2Q^{-5/2}Q'''')\,\mathrm{d}r. \tag{2.11}$$

Here a prime denotes differentiation. These expressions when used in (2.7) give the quantisation condition in the fourth order. We shall take up in § 3 the explicit evaluation of the I's for the quartic oscillator.

3. Application to the quartic oscillator

In the case of the quartic oscillator defined by the potential $V(r) = r^4$, the JWKB integrals I_0, I_2 and I_4 can be expressed in terms of complete elliptic integrals. The classical turning points r_1 and r_2 are the positive roots of the equation

$$W - r^4 - L^2/2r^2 = 0.$$

For this problem it proves to be convenient to define the following new quantities

$$z = W^{-1/2}r^2, \qquad R(z) = -z^3 + z - \sigma, \tag{3.1}$$

with

$$\sigma = L^2/2W^{3/2}. \tag{3.2}$$

The turning points $z = a$ and $z = b$ are then determined by the equation $R(z) = 0$, whose roots are

$$a = \frac{2}{\sqrt{3}}\cos\frac{\phi}{3}, \qquad b = \frac{2}{\sqrt{3}}\cos\frac{\phi + 4\pi}{3}, \qquad c = \frac{2}{\sqrt{3}}\cos\frac{\phi + 2\pi}{3}, \tag{3.3}$$

with $\cos\phi = -3\sqrt{3}\sigma/2$. It is easily checked that, for physical values of W, $a > b > 0 > c$.

3.1. Evaluation of I_0

The lowest-order integral I_0 has been shown in Seetharaman *et al* (1982) to be expressible as

$$I_0 = \sqrt{2}\,W^{3/4}[(\tfrac{2}{3} - \sigma/c)gK + \sigma(c^{-1} - b^{-1})g\Pi]. \tag{3.4}$$

In this expression $K \equiv K(k^2)$ and $\Pi \equiv \Pi(\alpha^2, k)$ are complete elliptic integrals of the first and the third kinds, in the notation of Byrd and Friedman (1971). Further

$$g = 2(a - c)^{-1/2}, \qquad k^2 = (a - b)/(a - c), \qquad \alpha^2 = ck^2/b. \tag{3.5}$$

3.2. Evaluation of I_2

In terms of z and R, the expression (2.10) for I_2 becomes

$$I_2 = (\sqrt{2}/32)\,W^{-3/4}\oint \mathrm{d}z\,R^{-3/2}(2z^2 + \sigma/z)$$

which can be written as

$$I_2=\frac{\sqrt{2}}{16}W^{-3/4}\left(2\frac{\partial}{\partial\sigma}\oint dz\, z^2R^{-1/2}+\sigma\frac{\partial}{\partial\sigma}\oint\frac{dz}{z}R^{-1/2}\right)$$
$$=\frac{\sqrt{2}}{16}W^{-3/4}\left(\frac{2}{3}\frac{\partial}{\partial\sigma}\oint dz\, R^{-1/2}+\sigma\frac{\partial}{\partial\sigma}\oint\frac{dz}{z}R^{-1/2}\right).$$

The last step is obtained by dropping a total derivative. Noting that the above integrands have only integrable singularities at the turning points, the contour of integration can be deformed until it coincides with the upper and lower lips of the cut. We then get

$$\oint dz\, R^{-1/2}=2\int_b^a dx\, R^{-1/2}=2gK$$

and

$$\oint\frac{dz}{z}R^{-1/2}=2\int_b^a\frac{dx}{x}R^{-1/2}=2g\left(\frac{K}{c}+(b^{-1}-c^{-1})\Pi\right).$$

Substituting these, we have the following expression for I_2:

$$I_2=\sqrt{2}\,W^{-3/4}\left[\frac{1}{12}\frac{\partial}{\partial\sigma}(gK)+\frac{1}{8}\sigma\frac{\partial}{\partial\sigma}\left(\frac{gK}{c}+g(b^{-1}-c^{-1})\Pi\right)\right]. \tag{3.6}$$

3.3. Evaluation of I_4

It is convenient to write

$$I_4=I_{41}+I_{42}$$

where

$$I_{41}=-\tfrac{7}{6144}\sqrt{2}\oint dr\, Q^{-7/2}(Q'')^2, \qquad I_{42}=\tfrac{2}{6144}\sqrt{2}\oint dr\, Q^{-5/2}Q''''.$$

In terms of z and $R(z)$ defined earlier, these integrals can be simplified as follows:

$$I_{41}=-\tfrac{21}{256}\sqrt{2}\,W^{-9/4}\oint dz[-\tfrac{4}{3}R^{-5/2}+(z-\sigma)R^{-7/2}+(\sigma^2/4z)R^{-7/2}]$$
$$=\sqrt{2}\,W^{-9/4}\left[\left(\frac{7}{48}\frac{\partial^2}{\partial\sigma^2}+\frac{7}{160}\sigma\frac{\partial^3}{\partial\sigma^3}\right)\oint dz\, R^{-1/2}-\frac{7}{160}\frac{\partial^3}{\partial\sigma^3}\oint dz\, zR^{-1/2}\right.$$
$$\left.-\frac{7}{160}\sigma^2\frac{\partial^3}{\partial\sigma^3}\oint\frac{dz}{z}R^{-1/2}\right].$$

As before, we have dropped some total derivative terms, and introduced derivatives with respect to σ so that the integrands have only integrable singularities at the turning points. By a similar procedure I_{42} can be reduced to the form

$$I_{42}=-\sqrt{2}\,W^{-9/4}\left(\frac{1}{576}\frac{\partial^2}{\partial\sigma^2}\oint dz\, R^{-1/2}+\frac{5}{192}\sigma\frac{\partial^2}{\partial\sigma^2}\oint\frac{dz}{z}R^{-1/2}\right).$$

Adding I_{41} and I_{42}, and expressing the closed contour integrals as integrals along the real axis from b to a, we get

$$I_4=\sqrt{2}\,W^{-9/4}\left[\left(\frac{83}{288}\frac{\partial^2}{\partial\sigma^2}+\frac{7}{80}\sigma\frac{\partial^3}{\partial\sigma^3}\right)gK-\frac{7}{80}\frac{\partial^3}{\partial\sigma^3}[gcK+g(a-c)E]\right.$$
$$\left.+\left(-\frac{5}{96}\sigma\frac{\partial^2}{\partial\sigma^2}+\frac{7}{20}\sigma^2\frac{\partial^3}{\partial\sigma^3}\right)\left(\frac{gK}{c}+g(b^{-1}-c^{-1})\Pi\right)\right]. \tag{3.7}$$

Here $E=E(k^2)$ is the complete elliptic integral of the second kind.

All the integrals occurring in the fourth order of approximation have now been expressed in terms of complete elliptic integrals. Since the derivatives of K, E and Π are again expressible as combinations of K, E, Π, the entire RHS of (2.7) can be written as a linear combination of these. The resulting equation then determines W implicitly. While this equation can be solved numerically for the energies†, it is certainly of interest to consider how one can invert the highly implicit relation and obtain an explicit analytic expression for W. The inversion cannot of course be done exactly, but it proves possible to derive, by an approximate inversion, a simple formula which is found to work extremely well.

4. Analytic formula for the energy

In each of the integrals I_0, I_2 and I_4, the energy W occurs only in the combination $\sigma=L^2/2W^{3/2}$, apart from an overall factor. For fixed L, σ is small for large W. Since the JWKB method is expected to be good for large quantum numbers (which in this case implies large values of W), we base our inversion procedure on an expansion of the JWKB integrals in powers of σ. As I_4 has a $W^{-9/4}$ outside as a factor, we carry the expansion of I_0 and I_2 up to terms of order $W^{-9/4}$, in order to include contributions from the fourth-order integral.

We start with the expansion of the roots a, b, c. For small σ we get

$$a\sim 1-\tfrac{1}{2}\sigma-\tfrac{3}{8}\sigma^2-\tfrac{1}{2}\sigma^3,\qquad b\sim\sigma+\sigma^3,\qquad c\sim -1-\tfrac{1}{2}\sigma+\tfrac{3}{8}\sigma^2-\tfrac{1}{2}\sigma^3.$$

These lead to the following:

$$k^2\sim\tfrac{1}{2}-\tfrac{3}{4}\sigma(1+\tfrac{11}{8}\sigma^2),\qquad \alpha^2\sim -(1/2\sigma)(1-\sigma-\tfrac{17}{8}\sigma^2+\mathrm{O}(\sigma^4)),$$
$$g\sim\sqrt{2}(1+\tfrac{3}{16}\sigma^2+\mathrm{O}(\sigma^4)).$$

The expansions of K, E and Π are given by

$$K(k^2)\sim\bar{K}-\tfrac{3}{4}(2\bar{E}-\bar{K})\sigma+\tfrac{9}{32}\bar{K}\sigma^2-\tfrac{213}{128}(2\bar{E}-\bar{K})\sigma^3$$
$$E(k^2)\sim\bar{E}-\tfrac{3}{4}(\bar{E}-\bar{K})\sigma+\tfrac{9}{32}(2\bar{K}-3\bar{E})\sigma^2+\tfrac{3}{128}(38\bar{K}-71\bar{E})\sigma^3,$$
$$\Pi(\alpha^2,k)\sim\pi\sqrt{\sigma}/\sqrt{2}-2(\bar{E}-\bar{K})\sigma-\pi\sigma\sqrt{\sigma}/\sqrt{2}-(\tfrac{4}{3}\bar{K}-\tfrac{1}{2}\bar{E})\sigma^2,$$

where

$$\bar{K}\equiv K(k^2=\tfrac{1}{2})=1.854\,074\,67, \tag{4.1}$$

and, by Legendre's relation,

$$2\bar{E}-\bar{K}=\tfrac{1}{2}\pi(\bar{K})^{-1}. \tag{4.2}$$

† In such a calculation, the value of L appropriate to the fourth order should be used (cf § 2).

Substituting these into the expressions for I_0, I_2 and I_4 given in § 3, we get the following expansions:

$$I_0 \sim W^{3/4}[\tfrac{4}{3}\bar{K} - \pi\sqrt{2\sigma} + (2\bar{E} - \bar{K})\sigma - \tfrac{5}{24}\bar{K}\sigma^2],$$
$$I_2 \sim W^{-3/4}[-\pi/8\sqrt{2\sigma} - \tfrac{1}{8}(2\bar{E} - \bar{K}) + \tfrac{25}{96}\bar{K}\sigma],$$
$$I_4 \sim W^{-9/4}[\pi/128(2\sigma)^{3/2} + \tfrac{11}{1536}\bar{K}].$$

On substituting for σ and adding like powers of W, the quantisation condition (2.7) becomes

$$\pi(2n_r+1) = -\pi(L + 1/8L - 1/128L^3) + \tfrac{4}{3}\bar{K}W^{3/4} + \tfrac{1}{2}(2\bar{E} - \bar{K})(L^2 - \tfrac{1}{4})W^{-3/4} + (\tfrac{11}{1536} + \tfrac{25}{192}L^2 - \tfrac{5}{96}L^4)\bar{K}W^{-9/4} + \mathrm{O}(W^{-15/4}). \tag{4.3}$$

Note the presence of W-independent terms in (4.3) which come from each of the integrals I_0, I_2 and I_4. Now, it is not difficult to convince oneself that higher-order integrals (when included) contribute to (4.3) as follows: the odd-order integrals I_{2n+3} vanish, and the even-order integrals are of the form

$$I_{2n} = -\pi L(4L^2)^{-n}\binom{\frac{1}{2}}{n} + \mathrm{O}(W^{-(2n-1)}).$$

As shown in I, the constant terms in I_{2n} are independent of the nature of the potential so long as $r^2V(r) \to 0$ as $r \to 0$ and their sum to all orders of the approximation is $-\pi L(1 + 1/4L^2)^{1/2}$ which must be set equal to $-\pi(l + \frac{1}{2})$. This will be the constant term in (4.3) if contributions from *all* orders are included. Further, if terms lower than $W^{-9/4}$ are neglected, the only other change in the RHS of (4.3) when all orders are summed will be the replacement of L^2 by $l(l+1)$. We thus get *exact* expressions for the first four terms on the RHS of (4.3):

$$\begin{aligned}\pi(2n_r+1) &= \tfrac{4}{3}\bar{K}W^{3/4} - \pi(l + \tfrac{1}{2}) + \tfrac{1}{2}(2\bar{E} - \bar{K})[l(l+1) - \tfrac{1}{4}]W^{-3/4} \\ &\quad + [\tfrac{11}{1536} + \tfrac{25}{192}l(l+1) - \tfrac{5}{96}l^2(l+1)^2]\bar{K}W^{-9/4} + \dots \\ &\equiv AW^{3/4} + B + CW^{-3/4} + DW^{-9/4} + \mathrm{O}(W^{-15/4}).\end{aligned} \tag{4.4}$$

The values of A, B, C and D in the case $l = 0$ are in agreement with those given by Bender *et al* (1977) for the one-dimensional quartic oscillator.

The relation (4.4) is the basis for our analytical formula for the energy levels of the quartic oscillator in three dimensions. It follows from (4.4) that W can be written in the form

$$W = a_1(n + \tfrac{3}{2})^{4/3}[1 + a_2(n + \tfrac{3}{2})^{-2} + a_3(n + \tfrac{3}{2})^{-4} + \dots] \tag{4.5}$$

where $n = 2n_r + l$. A simple calculation shows that the a_i, determined by putting (4.5) in (4.4), can be expressed in terms of $\bar{K}$ and l alone. The values are

$$a_1 = (3\pi/4\bar{K})^{4/3}, \qquad a_2 = (9\pi)^{-1}[1 - 4l(l+1)],$$
$$a_3 = -(5/81\pi^4)[\tfrac{1}{8}\pi^2 + \tfrac{11}{30}\bar{K}^4 + (\tfrac{20}{3}\bar{K}^4 - \pi^2)l(l+1) + (2\pi^2 - \tfrac{8}{3}\bar{K}^4)l^2(l+1)^2]. \tag{4.6}$$

The relations (4.5) and (4.6) provide an explicit analytical formula for the energy eigenvalues. The values of W predicted by this formula for different n and l values are given in table 1. The results are seen to be in excellent agreement with the highly accurate (to 1 part in 10^{15}) numerical eigenvalues of Bhargava (1982).

Table 1. Comparison of the JWKB energy values (equation (4.5)) with the exact energies for the three-dimensional quartic oscillator.

n	l	W_{JWKB}†	W_{exact} (Bhargava 1982)
0	0	2.399	2.393 644
1	1	4.4782	4.478 039
2	0	7.335 75	7.335 730
	2	6.829	6.830 308
5	1	16.599 528	16.599 521
	3	16.0461	16.046 193
	5	15.085	15.081 647
10	0	35.740 314	35.740 315
	4	34.980 19	34.980 152
	10	31.71	31.690 628
50	0	263.750 914	263.750 919
	20	257.891	257.889 588
	50	229.7	229.437 335

† The values are given only up to the decimal place where they begin to deviate from the exact ones.

5. Application to quartic oscillator in d dimensions

The quartic oscillator in d dimensions is characterised by the potential

$$V(r)=r^4=\left(\sum_{i=1}^{d} x_i^2\right)^2 .$$

The reduced Schrödinger equation in the variable r is

$$\mathrm{d}^2u/\mathrm{d}r^2+2\{W-r^4-[l+\tfrac{1}{2}(d-3)][l+\tfrac{1}{2}(d-1)]/2r^2\}u=0$$

(Hioe 1978). It is therefore clear that our analysis will go through for d dimensions with the trivial change of l to $l+\frac{1}{2}(d-3)$ which induces the change $n+\frac{3}{2}\to n+d/2$. With these replacements, (4.5) and (4.6) give the energy levels of the d-dimensional quantic oscillator. In table 2 we present some results for the case $d=2$.

6. Discussion

It is clear from our analysis that higher-order JWKB integrals for the pure quartic oscillator can be expressed in terms of complete elliptic integrals. In this work we have given the explicit expressions of these integrals in the fourth order of the JWKB approximation. Our analysis is based on the modified effective potential method, the details of which were outlined in an earlier paper (Seetharaman and Vasan 1984). The fact that all the JWKB integrals are closed contour integrals enables us to simplify the integrals considerably (by dropping total derivatives from integrands, etc) before they are expressed as convergent real integrals. Had the higher-order integrals been taken as real integrals to start with, suitable regularisations would have been necessary (see e.g. Pasupathy and Singh 1981 for S-waves). The results for the three-dimensional

Table 2. Comparison of the JWKB energy values with the exact energies for the two-dimensional quartic oscillator.

n	l	W_{JWKB}	W_{exact} (Bhargava 1982)
0	0	1.485	1.477 150
1	1	3.400	3.398 150
2	0	6.0032	6.003 386
	2	5.6235	5.624 339
5	1	14.977 82	14.977 808
	3	14.5085	14.508 675
	5	13.603	13.600 878
10	0	33.694 277	33.694 280
	4	33.066 979	33.066 978
	10	29.92	29.899 842
50	0	260.345 80	260.345 813
	20	254.727	254.725 806
	50	226.7	226.484 799

oscillator are easily extended to d dimensions by a suitable change of the parameters n and l.

The final formula for the energy levels yields very good results even for low values of n, the error being only about 0.2% even for the ground state. For levels with $n = l$, the results are again found to be good, although the parameter σ is not too small ($\sim$0.3) for these levels. A consequence of our formula is that the energy decreases with l for a given n. For fairly low values of l, the splitting is proportional to $l(l+1)$ for $d=3$, and to l^2 for $d=2$. This point was noted by Bell *et al* (1970) from their numerical results.

The higher-order JWKB analysis of radial problems can also be carried out by a different method advocated by Krieger and Rosenzweig (1967). In this method the Langer transformation is first applied to the radial equation so that the problem becomes truly one-dimensional, with the new independent variable ranging from $-\infty$ to $+\infty$. The method of Dunham is then applied to this one-dimensional equation to write down the higher-order JWKB integrals occurring in the quantisation condition. These integrals are then re-expressed in terms of the radial variable r. In this formalism, the centrifugal barrier parameter $l(l+1)$ is transformed to $(l+\frac{1}{2})^2$ and remains the same in every order. The expressions for the higher-order corrections are rather complicated, and they cannot be written down in terms of an effective potential. Regarding the equivalence of the two methods, the following observations may be noted. If only the lowest-order approximation is considered, the two methods yield identical results for any potential. When higher orders are included, it is not obvious whether the two methods are equivalent. In the context of the quartic oscillator, we find that the higher-order integrals yield different expressions, which shows that the energies (when numerically calculated) will be slightly different. This point has been noted earlier by Fröman and Fröman (1974). However, when an asymptotic expansion for the energy is made, the method of Krieger and Rosenzweig yields a series which is identical to the one defined by (4.5) and (4.6). The details of this calculation are outlined in the appendix.

Appendix. The method of Krieger and Rosenzweig

In the method of Krieger and Rosenzweig (1967), the energy eigenvalues are determined by quantisation condition which, in the fourth order JWKB approximation for $V(r)=r^4$, reads

$$(2n_r+1)\pi = J_0+J_2+J_4$$

where

$$J_0=\sqrt{2}\oint \mathrm{d}r\, G^{1/2}/r,$$

$$J_2=-(1/32\sqrt{2})\oint \mathrm{d}r\, r(\mathrm{d}G/\mathrm{d}r)^2 G^{-5/2},$$

$$J_4=\frac{-\sqrt{2}}{8192}\oint \mathrm{d}r\, r^3\left[49\left(\frac{\mathrm{d}G}{\mathrm{d}r}\right)^4 G^{-11/2}-16r\frac{\mathrm{d}G}{\mathrm{d}r}G^{-7/2}\left(\frac{r\mathrm{d}}{\mathrm{d}r}\right)^3 G\right],$$

with

$$G=r^2(W-r^4)-\tfrac{1}{2}(l+\tfrac{1}{2})^2.$$

The contour of integration in the above is the same as that described earlier.

The integral J_0 is trivially written down, as it is the same as I_0, (2.8), except that L^2 in I_0 should be replaced by $(l+\frac{1}{2})^2$. J_2 and J_4 can be considerably simplified by dropping total derivatives from the integrands. After a suitable change of variable and some labour, J_2 and J_4 can be expressed as

$$J_2=(W^{-3/4}/6\sqrt{2})(\partial/\partial\rho)\oint \mathrm{d}z\, S^{-1/2},$$

$$J_4=\frac{-\sqrt{2}}{64}W^{-9/4}\left(-\frac{1616}{45}\frac{\partial^2}{\partial\rho^2}\oint \mathrm{d}z\, S^{-1/2}+\frac{56}{5}\frac{\partial^3}{\partial\rho^3}\oint \mathrm{d}z\, z\, S^{-1/2}-\frac{224}{15}\rho\frac{\partial^3}{\partial\rho^3}\oint \mathrm{d}z\, S^{-1/2}\right)$$

where

$$\rho=(l+\tfrac{1}{2})^2/2W^{3/2}, \qquad S(z)=-z^3+z-\rho.$$

All the above integrals are expressible in terms of complete elliptic integrals. Expanding the latter integrals in powers of ρ one obtains

$$J_0\sim W^{3/4}[-\pi\sqrt{2\rho}+\tfrac{4}{3}\bar{K}+(2\bar{E}-\bar{K})\rho-\tfrac{5}{24}\bar{K}\rho^2],$$

$$J_2\sim W^{-3/4}[-\tfrac{1}{4}(2\bar{E}-\bar{K})+\tfrac{5}{16}\bar{K}\rho],$$

$$J_4\sim W^{-9/4}(-\tfrac{11}{384}\bar{K}).$$

Adding these, the quantisation condition becomes

$$\pi(2n_r+1)=\tfrac{4}{3}\bar{K}W^{3/4}-\pi(l+\tfrac{1}{2})+\tfrac{1}{2}[-\tfrac{1}{2}+(l+\tfrac{1}{2})^2](2\bar{E}-\bar{K})W^{-3/4}$$
$$+[-\tfrac{11}{384}+\tfrac{5}{32}(l+\tfrac{1}{2})^2-\tfrac{5}{96}(l+\tfrac{1}{2})^4]\bar{K}W^{-9/4}+\ldots.$$

It is easily checked that this is identical to (4.4). Consequently, one will get the same asymptotic series for the energy as the one described in the text ((4.5) and (4.6)).

References

Bell S, Davidson R and Warsop P A 1970 *J. Phys. B: At. Mol. Phys.* **3** 123
Bender C M, Olaussen K and Wang P S 1977 *Phys. Rev.* D **16** 1740
Bhargava V T A 1982 *PhD Thesis* (Madras University) unpublished
Byrd P F and Friedman M D 1971 *Handbook of Elliptic Integrals for Engineers and Scientists* 2nd edn (Berlin: Springer)
Dunham J L 1932 *Phys. Rev.* **41** 713
Fröman N and Fröman P O 1974 *Nuovo Cimento* B **20** 121
Hioe F T 1978 *J. Chem. Phys.* **69** 204
Krieger J B and Rosenzweig C 1967 *Phys. Rev.* **164** 171
Mathews P M, Seetharaman M and Sekhar Raghavan 1982 *J. Phys. A: Math. Gen.* **15** 103
Pasupathy J and Singh V 1981 *Z. Phys.* C **10** 23
Seetharaman M, Raghavan S and Vasan S S 1982 *J. Phys. A: Math. Gen.* **15** 1537
Seetharaman M and S S Vasan 1984 *J. Phys. A: Math. Gen.* **17** 2485